

# A Superglass Phase of Interacting Bosons

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We introduce a Bose-Hubbard Hamiltonian with random disordered interactions as a model to study the interplay of superfluidity and glassiness in a system of three-dimensional hard-core bosons at half-filling. Solving the model using large-scale quantum Monte Carlo simulations, we show that these disordered interactions promote a stable superglass phase, where superflow and glassy density localization coexist in equilibrium without exhibiting phase separation. The robustness of the superglass phase is underlined by its existence in a replica mean-field calculation on the infinite-dimensional Hamiltonian.

Despite the simplicity of its constituent atoms, the phase diagram of bulk helium has proven to be compelling and controversial – especially in light of recent observations of supersolid behavior in the crystal phase of Helium-4 (<sup>4</sup>He) [1, 2]. Several experiments have shown that superflow is enhanced by imperfections in the crystal lattice, such as roaming defects and interstitial atoms [3]. Most strikingly, new experiments by Davis and collaborators indicate that the onset of superflow is intimately tied to relaxation dynamics characteristic of glassy, or amorphous, solids [4]. The precise relationship between superflow and glassiness, and the possible existence of a novel *superglass* (SuG) state, is not without controversy in the experimental community [5]. Even outside of the <sup>4</sup>He context, the possible existence of a bosonic superglass phase is of broad theoretical importance – one that can be explored for example in simple models of interacting bosons. Lattice models have been instrumental in explaining some of the basic phenomenology of strongly-coupled quantum systems, including that of <sup>4</sup>He. The prototypical model is the lattice Bose-Hubbard (BH) model [6]. Its extended phase diagram is known to contain superfluids, Mott-insulating “crystals”, and supersolid phases, where the latter refers to a state with coexisting superfluidity and broken translational symmetry in the particle density [7].

Models of interacting bosons with disorder have been considered for some time [6, 8] typically in the local chemical potential, as might be realized in <sup>4</sup>He absorbed in porous media. In these and related models [9] arises either a “Bose-glass” (BG) phase, with localized disorder in the particle densities but no coexisting superflow, or superfluid phases with locally inhomogeneous superflow [10] – neither of which correspond to a SuG state. This can be understood in part by considering the nature of the states in the BG close to the Mott lobes which, through Anderson localization [11], essentially form single-particle localized states. For this reason, the BG cannot support phase coherence; other types of interactions beyond the random local chemical potential are necessary to induce superglassiness. In this paper we show that a thermodynamic SuG state of bosons can be

stabilized via random pairwise boson-boson interactions.

It is widely believed that disorder (in the form of dislocations, grain boundaries, impurities, etc.) plays a role in the supersolid behavior of <sup>4</sup>He [3], although the precise nature of this role is controversial. At the microscopic level, <sup>4</sup>He atoms interact via a deterministic pairwise potential in the continuum. If perfect crystal order is avoided by an effective quenching [12] into a glassy phase at low temperature, even without explicit randomness in the Hamiltonian, both disorder and random frustration can be self-generated from the deterministic potential, similar to scenarios put forward in discussions of the structural glass transition [13]. Therefore, we propose that an effective theory to describe glassy behavior in <sup>4</sup>He via this disordered and frustrated environment is a BH lattice model with *randomly frustrated* boson-boson interactions. We note that the phenomenology of random frustration and glassiness can also be constructed via a mathematical embodiment of a BH model on a random graph [14] – an approach that leaves open the question of whether a SuG exists for real finite dimensional lattices.

While the relevance of our model to the physics of <sup>4</sup>He ultimately relies on microscopic details via which disorder and frustration may be self-generated, it is of direct pertinence to collective phenomena in ultracold atomic gases. There, disorder may be introduced through mechanisms such as speckle potentials or superimposed optical lattices with incommensurate periods [15].

Accordingly, we introduce the hard-core BH model,

$$H = - \sum_{\langle ij \rangle} V_{ij} (n_i - 1/2)(n_j - 1/2) - t \sum_{\langle ij \rangle} (b_i^\dagger b_j + b_i b_j^\dagger), \quad (1)$$

where  $\langle ij \rangle$  indicates nearest-neighbor lattice sites. Here,  $n_i$  is the density operator of bosons hopping on a three-dimensional (3D) cubic lattice,  $b_i$  ( $b_i^\dagger$ ) is the annihilation (creation) operator.  $V_{ij}$  is a quenched random potential with bimodal distribution given by  $P(V_{ij}) = [\delta(V_{ij} - V) + \delta(V_{ij} + V)]/2$ , reminiscent of the canonical spin glass (SG) models studied extensively [16, 17]. The hopping ( $t$ ) term is the standard kinetic energy term in the BH model, acting as a source of quantum fluctua-

tions which favor a superfluid phase whenever  $V/t$  is sufficiently small. Eq. (1) is explicitly defined to maintain half-filling. Below, we use large-scale Stochastic Series Expansion (SSE) quantum Monte Carlo (QMC) simulations to demonstrate that these simple random disordered interactions can promote a stable superglass phase in the Hamiltonian (1), where superflow and glassy density localization coexist in equilibrium without exhibiting phase separation. Following that, the existence of this superglass phase is corroborated by a replica mean-field calculation on an infinite-dimensional model.

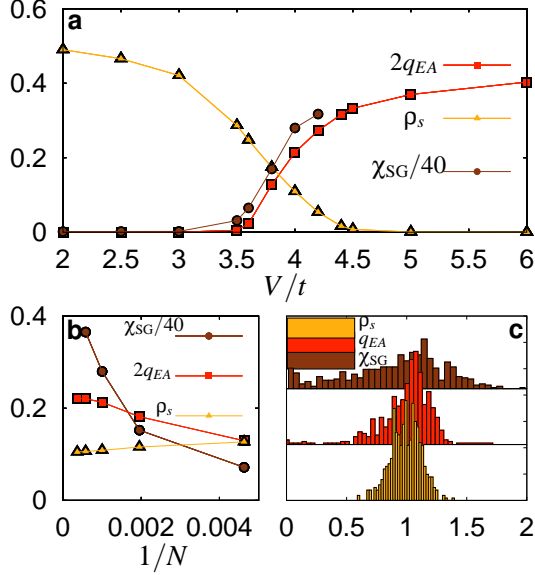


FIG. 1: **a)** The superfluid density, EA order parameter, and spin glass susceptibility as a function of  $V/t$ , for  $T/t = 1/4$  and  $L = 10$ . For  $V/t > 4.2$ ,  $\chi_{SG}$  suffers from algorithmic freezing or loss of ergodicity, making results unreliable without a refinement of simulation technique (e.g. parallel tempering). **b)** Finite-size scaling of the QMC observables in the SuG state at  $V/t = 4$  and  $T/t = 1/4$ . In **(a)** and **(b)**, statistical error bars related to QMC sampling are smaller than the symbol size. **c)** Histogram of the three observables at  $L = 10$  in **(b)** using 500 RODs. The x-axis has been divided by the mean of the distribution in each case, and the y-axis of each has been offset for clarity. There is no clear separation of the SuG RODs into distinct glassy and superfluid bins.

QMC simulations are performed using the finite-temperature SSE technique [18], modified to allow for the random disorder in Eq. (1) by employing different directed loop equations for each individual bond interaction. A single simulation consists of constructing a 3D cubic lattice (size  $N = L^3$ ) with some unique choice of specific bimodal bond distribution,  $V_{ij}$ , referred to as one realization of disorder (ROD). Each specific simulation, for a given system size, temperature, and interaction strength  $V/t$ , is equilibrated for  $10^6$  MC steps before thermodynamic data is collected (for  $2 \times 10^6$  steps), this MC average denoted below by  $\langle \dots \rangle$ . The simulation pro-

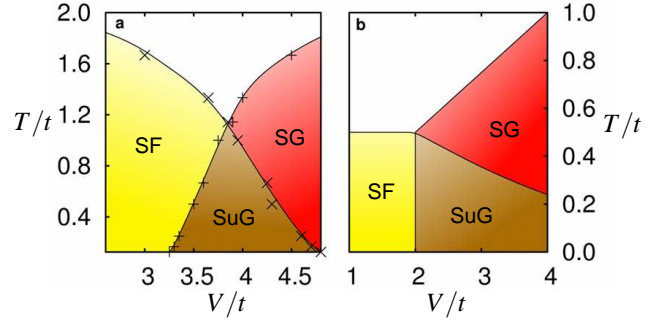


FIG. 2: The phase diagram of the **(a)** quantum (Eq. 1) and **(b)** mean-field (Eq. 7) models, solved with QMC and replica mean-field methods respectively. Both phase diagrams show a robust superglass (SuG) phase, as well as superfluid (SF) and quantum spin glass (SG) states. The data points in **a**, used to determine the phase boundaries, were obtained when  $\rho_s$  and  $q_{EA}$  surpassed an approximate numerical value of  $1/\sqrt{N}$  on a  $N = 12 \times 12 \times 12$  system. It is interesting to note that the main qualitative difference between the two phase diagrams is the absence of a direct superfluid to superglass transition in the mean-field solution, **(b)**, as a function of temperature.

cedure is repeated for many RODs – typically between  $10^2$  and  $10^3$ , and denoted as  $[\dots]_{\text{avg}}$ .

We collect data for thermodynamic quantities which are able to quantify both the superfluid and the glassy characteristics of various phases of the model. Our definition for superflow is based on the superfluid density,

$$\rho_s = \frac{1}{3\beta N t} \left[ \sum_{k=x,y,z} \langle W_k^2 \rangle \right]_{\text{avg}} \quad (2)$$

where  $W_k$  is the winding number [19] measured in the lattice direction  $k$ . Two quantities commonly used in SG models to indicate a glassy phase are the Edwards-Anderson (EA) order parameter,

$$q_{EA} = \frac{1}{N} \left[ \sum_i \langle (n_i - 1/2)^2 \rangle \right]_{\text{avg}}, \quad (3)$$

and the replica overlap parameter [16],

$$q_{ab} = \frac{1}{N} \sum_i (n_i - 1/2)^a (n_i - 1/2)^b, \quad (4)$$

where two identical (independently simulated) replicas of the system are labeled  $a$  and  $b$  for a single ROD. The disorder-averaged SG susceptibility is thus defined as

$$\chi_{SG} = N [\langle q_{ab}^2 \rangle]_{\text{avg}}. \quad (5)$$

Figure 1 **a** shows these observables as a function of  $V/t$ . For some finite region of  $V/t$ , the model displays robust superflow (characterized by finite  $\rho_s$ ) and a finite EA order parameter and SG susceptibility – these trends survive into the thermodynamic limit (see Fig. 1 **b**). This

indicates that the SuG is a stable equilibrium phase in this model, persisting down to zero temperature (Fig. 2).

The simplicity of Hamiltonian (1) suggests a clear mechanism for the formation of the SuG phase; it arises from the competition of disorder in the boson interactions with quantum fluctuations. One may therefore wonder whether this mechanism is robust enough to be found in standard analytical treatments of well-studied spin glass models. We address this question by employing an exact mapping of the hard-core boson model in Eq. (1) to a quantum spin model through the transformation to spin-1/2 operators:  $S_i^z = n_i - 1/2$ ,  $S_i^- = b_i$ , and  $S_i^+ = b_i^\dagger$ . Equation 1 can then be written as a standard XXZ model:

$$H = - \sum_{\langle ij \rangle} V_{ij} S_i^z S_j^z - J_{xy} \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y), \quad (6)$$

where  $J_{xy} = 2t$  and the off-diagonal part of the Hamiltonian is the XY model, which (e.g. for  $V = 0$ ) has in-plane ferromagnetic long-range order,  $\langle S^x \rangle \neq 0$ , corresponding to the superfluid state in the boson language. One now recognizes the diagonal part of (6) as the standard EA Ising spin model, for which the expected SG phase is reproduced for our model as  $t \rightarrow 0$  (Figs 1 and 2).

An analytical solution for the 3D EA model ( $t = 0$ ) is not available. However, the mean-field (MF) Sherrington-Kirkpatrick model can be solved exactly by Parisi's replica symmetry breaking (RSB) ansatz [20]. Here, we generalize the replica MF method to include the effects from both the conventional  $V_{ij}$  Ising coupling, and the off-diagonal  $J_{xy}$ -coupling [21]. The MF model is

$$H_{MF} = - \sum_{ij} V_{ij} S_i^z S_j^z - \frac{J_{xy}}{N} \sum_{ij} (S_i^x S_j^x + S_i^y S_j^y), \quad (7)$$

where the summation is over all distinct pairs of spins. For simplicity, we use a Gaussian distribution,  $P(V_{ij}) = \frac{1}{\sqrt{2\pi V^2/N}} \exp\left(\frac{-NV_{ij}^2}{2V^2}\right)$ , which does not qualitatively change the physics compared to a bimodal  $P(V_{ij})$ . The disorder averaged free energy using the replica formalism is given by  $\beta N f = -[\ln Z]_{\text{avg}} = -\lim_{n \rightarrow 0} \frac{1}{n} ([Z^n]_{\text{avg}} - 1)$ , where  $[\dots]_{\text{avg}}$  denotes disorder averaging. We introduce the Hubbard-Stratonovich (HS) fields for SG overlap  $Q^{ab} = \langle S_z^a S_z^b \rangle \forall a \neq b$ , overlap within the same replica  $R^a = \langle S_z^a S_z^a \rangle$ , and off-diagonal ordering parameter  $M^a = \langle S_x^a \rangle$ . We assume that  $M^a = M$  and  $R^a = R$  are replica independent and, for simplicity, consider solely a 1-step RSB for  $Q^{ab}$ . Following Parisi's parameterization for 1-step RSB [16], we divide the  $n$  replicas into  $n/m$  groups of  $m$  replicas, and set  $Q_{ab} = Q_1$  if  $a$  and  $b$  belong to the same group; or  $Q_{ab} = Q_0$  if  $a$  and  $b$  belong to different groups. Assuming that the HS fields are static, we obtain the free energy at 1-step RSB,

$$\beta f = -\frac{1}{4}(\beta V)^2 \left[ (R - Q_1)(1/2 - R - Q_1) \right.$$

$$\left. -m(Q_1^2 - Q_0^2) \right] + \frac{1}{2}\beta J_{xy} M^2 - \frac{1}{m} \int Dx \ln \left[ 2 \int Dy \cosh^m(\beta/2 \sqrt{H_z^2 + (J_{xy} M)^2}) \right], \quad (8)$$

where  $\int Dx F(x) \equiv (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} dx \exp(-x^2/2) F(x)$ , and  $H_z = V(x\sqrt{Q_0} + y\sqrt{Q_1 - Q_0})$ . The replica symmetric free energy can be recovered by setting any one of the conditions  $Q_1 = Q_0$ ,  $m = 0$  or  $m = 1$ . The self-consistent MF equations are obtained by optimizing the free energy. We find four phases analogous to those in the hard-core boson model (see Fig. 2). In particular, a SuG phase, defined by coexisting in-plane order  $M \neq 0$  (the magnetic analog to the superfluid order parameter), and SG overlap parameters  $Q_0 \neq 0$  and  $Q_1 \neq 0$  is clearly obtained. The vertical nature of the SuG-SF phase boundary originates from the observation that the replica theory does not admit a self-consistent set of equations with nonzero  $M$  and  $Q_0$  when  $V/t \leq 1$ . Physically, this results from the absence of phase fluctuations in the superfluid phase when the SG order parameter  $Q_0$  is nonzero.

In conclusion, both large-scale QMC simulations and replica MF calculations exhibit a robust, stable, equilibrium superglass phase in the BH model, Eq. (1). The simplicity of the model reveals that the SuG phase develops under competition between random disordered interactions and quantum fluctuations. QMC simulations allow us to characterize the SuG phase in great detail; for example, as evident from the single ROD illustrated in Fig. 3, there is no phase separation into distinct "super" and "glassy" regions within the simulation cell. Rather, the bulk of the particles in the system exhibit both off-diagonal superfluid and diagonal glassy ordering. Most importantly, our results clearly indicate that glassiness can co-exist with superfluidity in a general bosonic system in three dimensions. Although we did not directly study a microscopic model of  $^4\text{He}$ , in the event that a metastable SuG [12] does exist in  $^4\text{He}$  within experimentally-accessible timescales [4], quantum fluctuations and random frustration are the most relevant, if not inevitable, physical ingredients at play. Our model provides an affirmative answer that a SuG is possible under these two conditions. Furthermore, since the feasibility of obtaining disordered interactions similar to our  $V$  term [Eq. (1)] has recently been demonstrated in Fermi-Bose mixtures in random optical lattices [15], such cold atomic systems might also be suitable places to look for superglass phases in the future. With recent advances in both cold atom experiments and numerical methods [22], the broad class of disordered Hamiltonian introduced in this paper provides new experimental and theoretical motivation to study the interplay between superfluidity and quantum glassiness in numerous extensions of our model.

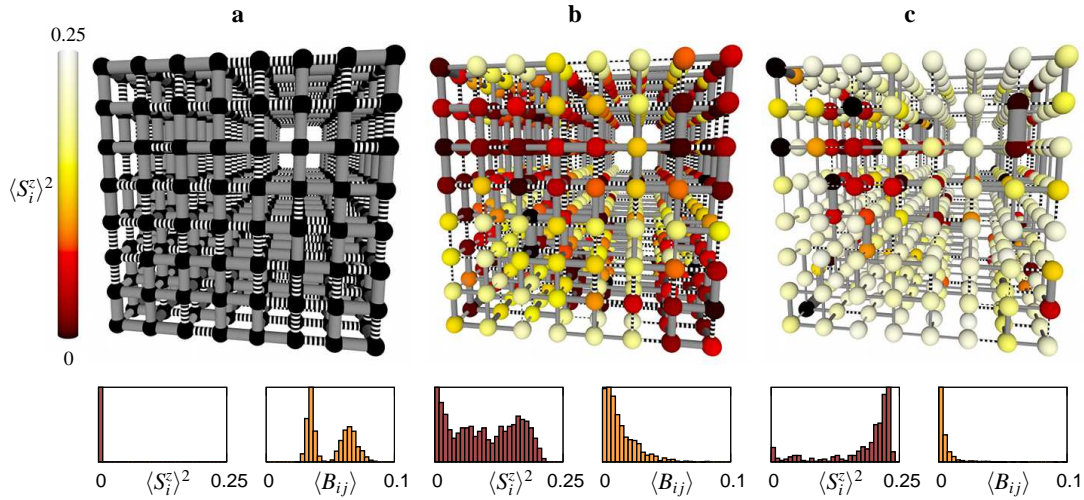


FIG. 3: The QMC simulation cell, with one random realization of disorder, averaged over  $2 \times 10^6$  Monte Carlo (MC) steps. Grey bonds correspond to  $V_{ij} = V$ , striped bonds to  $V_{ij} = -V$ . On the sites  $i$  of the lattice, the density localization is illustrated by the squared average of  $S_i^z = n_i - 1/2$ , which is black for a site where a boson is delocalized (a “superfluid site”), white for a localized (frozen) site, and colored for partially localized Boson densities ( $0 \leq \langle S_i^z \rangle^2 \leq 0.25$ ). The bond-strength, defined as the boson hopping operator  $\langle B_{ij} \rangle = \langle b_i^\dagger b_j + b_i b_j^\dagger \rangle$  is represented as the thickness of the bonds. At bottom, the histograms of  $\langle S_i^z \rangle^2$  (left), and  $\langle B_{ij} \rangle$  (right) for each case: superfluid (a) at  $V/t = 3$ , superglass (b) at  $V/t = 4$ , and spin glass (c) at  $V/t = 5$ . The double-peaked  $\langle B_{ij} \rangle$  histogram in (a) simply results from the bimodal distribution of  $V_{ij}$ .

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